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1. Oktober 2018

Algorithms & Data Structures — Homework 2 — HS 18

Exercise Class (Room & TA): ____________________

Submitted by: ____________________

Peer Feedback by: ____________________

Points: _____

Exercise 2.1 *Number of Edges and Subgraphs.*

Answer the following questions and justify your answers with a brief explanation.

1. How many edges does an undirected graph of n vertices maximally contain? How many edges does a directed graph of n vertices maximally contain? (In both cases, assume that the graph does not contain loops.)

   The number of edges in an undirected graph is at most $\binom{n}{2}$: an n-element ground set (vertices) contains exactly that many different 2-element subsets (edges). Another way to see this: every vertex can share an edge with each of the other $n-1$ vertices, and to prevent double counting, we need to divide by two. Hence $n \cdot (n-1) \cdot \frac{1}{2} = \binom{n}{2}$.

   In directed graphs, the number of edges is at most $n(n-1)$. There can be an edge from each of the n vertices pointing to one of the $n-1$ other vertices.

2. What is the maximum number of edges in an undirected k-partite graph with $n = \sum_{i=1}^{k} u_k$ vertices, where $u_i > 0$ is the number of vertices in the i-th subset of this partition?

   A vertex in the i-th subset can share an edge with all vertices not in the i-th subset, thus $(n-u_i)$ edges. This makes $u_i \cdot (n-u_i)$ for all vertices in this subset. Summing this up and prevent double counting (by dividing by 2) we get a total number of edges of

   $$\frac{1}{2}\sum_{i=1}^{k} u_i(n-u_i) = \frac{n^2 - \sum_{i=1}^{k} u_i^2}{2}\,.$$

3. Given an undirected clique G of size n, where n is an odd prime number. How many pairwise edge-disjoint simple cycles (i.e. cycles that use every vertex at most once) of length n does G contain?

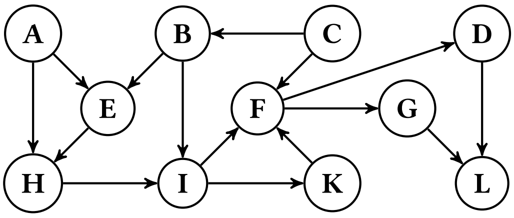
   The number of edges in G is $\binom{n}{2}$, and the number of edges in such a cycle is n. Hence, an upper bound on the number of edge-disjoint cycles is $\frac{n(n-1)}{2n} = \frac{n-1}{2}$.

   This bound is tight: Let the vertices be $v_0, \dots, v_{n-1}$. Consider the collection of cycles $C^{(i)} = (V, E^{(i)})$ with $E^{(i)} := \{\{v_j, v_k\} \colon (k-j) \equiv_n i\}$ for all $1 \le i \le \frac{n-1}{2}$. The sets of edges indeed form a cycle of length n: To see this, note that $1, \dots, \frac{n-1}{2}$ are all co-prime with n. Thus

$(v_0, v_i, v_{2i \mod n} \dots, v_{(n-1)i \mod n}, v_0)$ forms a simple cycle. Furthermore, the sets $E^{(i)}$ are pairwise disjoint because every edge belongs to exactly one set $E^{(i)}$: let $k > j$, then either $k-j \le \frac{n-1}{2}$ or $n + j - k \le \frac{n-1}{2}$. Hence, every edge is in at most one of these sets.

Exercise 2.2 *Topological Sorting* **(1 point)**.

1. How many topological orders does the following graph contain? List all topological orders.



Solution: The answer is 6:

(a) $A, C, B, E, H, I, K, F, D, G, L$

(b) $A, C, B, E, H, I, K, F, G, D, L$

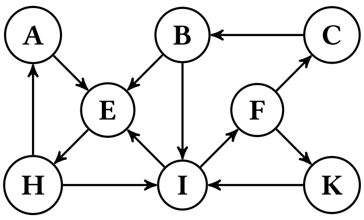
(c) $C, A, B, E, H, I, K, F, D, G, L$

(d) $C, A, B, E, H, I, K, F, G, D, L$

(e) $C, B, A, E, H, I, K, F, D, G, L$

(f) $C, B, A, E, H, I, K, F, G, D, L$

2. Consider now the following graph $G = (V, E)$ and find a set $E' \subset E$ of minimum cardinality, such that $G' = (V, E \setminus E')$ can be topologically sorted. Justify your answer (i.e., why is it necessary to remove at least $|E'|$ edges?).



Solution: From each directed cycle in G, at least on edge needs to be removed, such that G becomes acyclic. The cycles are (E, H, I), (E, H, A), (F, K, I), (I, F, C, B), and (I, F, C, B, E, H). Removing the edges (E, H) and (I, F) makes the graph indeed acyclic. It is not possible to remove less edges, since the two cycles (E, H, A) and (F, K, I) are edge-disjoint.

3. What is the maximum number of edges in a directed graph that can be topologically sorted? Formulate your claim for every positive integer n and prove your claim by using induction.

__Hint:__ For each $n \in \{1, 2, 3, 4\}$ construct a graph with n vertices that is topologically sortable and contains the maximum number of edges. Then use these observations to derive your claim.)

Solution: We conjecture, that an acyclic directed graph cannot contain more than $\sum_{i=1}^{n-1} i = n(n-1)/2$ edges. We shall prove this claim using induction on n.

Base Case.
Let $n = 1$. Every acyclic graph with one vertex has exactly (and thus also maximum) $0 = n(n-1)/2$ edges.

Induction Hypothesis.
Every acyclic graph with n vertices has at most $n(n-1)/2$ edges, for some positive integer n.

Indctive Step.
We must show that the property holds for $n+1$. Consider an arbitrary acyclic graph $G = (V, E)$ with $n+1$ vertices. Since the graph is acyclic, it must contain at least one vertex v with incoming degree 0 (shown in the lecture). When we remove this vertex and all incident edges from G, we get a new graph G' with n vertices. This new graph is also acyclic, since by removing edges no new cycles can appear. Using the induction hypothesis, G' has at most $n(n-1)/2$ edges. Furthermore, the vertex v that we removed from G had at most n incident edges. Thus, we removed at most n edges from G to obtain G'. Therefore, the original graph G contained at most

$$n + n(n-1)/2 = 2n/2 + n(n-1)/2 = n(n+1)/2$$

edges.

Note that this upper bound is tight: The graph G with vertex set $V = \{v_1 \ldots, v_n\}$ and edge set $E = \{(v_i, v_j) \colon 1 \leq i < j \leq n\}$ has exactly $\frac{n(n-1)}{2}$ edges and $(v_1 \ldots, v_n)$ is a topological order of G.

Exercise 2.3 *Number of Edges and Connected Components.*

1. Prove via mathematical induction that a connected graph with $n > 0$ vertices has at least $n-1$ edges.

   - **Base Case.**
     Let $n = 1$. Then a graph contains only 1 vertex and $0 = n - 1$ edges.

   - **Induction Hypothesis.**
     Assume that the property holds for every positive integer $l \leq k$. That is, every connected graph with $l \leq k$ vertices has at least $l - 1$ edges.

   - **Inductive Step.**
     We must show that the property holds for $k+1$. Consider a connected graph with $k+1$ vertices. Take any vertex v. Let's remove v from G and consider connected components $C_1 = (V_1, E_1), \ldots, C_m = (V_m, E_m)$ of the new graph. We can bound $m \leq \deg_G v$, because each component should contain at least one neighbour of v in G. Indeed, if some C_i does not contain any neighbour of v, there is no path from vertices of C_i to v in G, which contradicts the fact that G is connected. Since each C_i has $V_i < k+1$ vertices, by induction hypothesis $|E_i| \geq |V_i| - 1$. Hence the total number of edges in G is

$$\sum_{i=1}^{m} |E_i| + \deg_G v \geq \sum_{i=1}^{m} |V_i| - \sum_{i=1}^{m} 1 + \deg_G v = k - m + \deg_G v \geq k$$

   By the principle of mathematical induction, this is true for any positive integer n.

2. Prove that a graph G with n vertices and m connected components has at least $n - m$ edges.

   Solution: Let $C_1, \ldots, C_m$ be connected components of G. Since each $C_i = (V_i, E_i)$ is connected, $|E_i| \geq |V_i| - 1$. Hence the total number of edges in G is

$$\sum_{i=1}^{m} |E_i| \geq \sum_{i=1}^{m} |V_i| - \sum_{i=1}^{m} 1 = n - m.$$

   Recall that an undirected acyclic graph is called a *forest*. It is easy to see that a graph G is a forest iff each connected component of G a is a tree.

3. Prove that a forest G with n vertices and m connected components has $n - m$ edges.

   Solution: Let $T_1, \ldots, T_m$ be connected components of G. Since each $T_i = (V_i, E_i)$ is connected, $|E_i| = |V_i| - 1$. Hence the total number of edges in G is

$$\sum_{i=1}^{m} |E_i| = \sum_{i=1}^{m} |V_i| - \sum_{i=1}^{m} 1 = n - m.$$

4. Prove that if a graph G with n vertices and m connected components has $n - m$ edges, then G is a forest.

   Solution: Proof by contradiction: Assume that there is a graph G with n vertices and m connected components which has $n - m$ edges and is not a forest. For each component $C_i = (V_i, E_i)$ of G, $|E_i| \geq |V_i| - 1$, and if $|E_i| = |V_i| - 1$, this component is a tree. Since G is not a forest, it has a component which is not a tree, i.e. a component $C_1 = (V_1, E_1)$ such that $|E_1| \geq |V_1|$. Hence the total number of edges in G is at least

$$\sum_{i}^{m} E_i = |E_1| + \sum_{i=2}^{m} |E_i| \geq |V_1| + \sum_{i=2}^{m} |V_i| - \sum_{i=2}^{m} 1 = n - (m-1) > n - m$$

   which is a contradiction.

Exercise 2.4 *Hamiltonian paths in directed acyclic graphs* **(2 points)**.

A *Hamiltonian path* in a (directed or undirected) graph G is a path in G that visits each vertex of G exactly once. It is known that a problem of finding a Hamiltonian path in a graph is **NP**-hard, which means that it is highly unlikely that this problem can be solved in polynomial time. However, for special types of graphs it is possible to solve this problem efficiently.

For directed acyclic graphs one can find a Hamiltonian path using topological sorting. To show this, answer the following questions about topological orderings and Hamiltonian paths:

1. Let G be a directed acyclic graph which has a Hamiltonian path. What is the relationship between the set of Hamiltonian paths and the set of topological orderings of G? What are the sizes of these sets?

2. Let G be a directed acyclic graph with no Hamiltonian paths. Can G have a unique topological ordering?

Solution:

1. Consider some Hamiltonian path P. Let's show that any topological ordering of G coincides with P. Assume that there exists a topological ordering $v_1, \dots, v_n$ which differs from $P = u_1, \dots, u_n$. Let k be the first position where these sequences differ. It means that $u_1 = v_1, \dots, u_{k-1} = v_{k-1}$, but $u_k = v_l$ for some $l > k$ and $v_k = u_m$ for some $m > k$. Hence $u_k, \dots, u_m$ is a path in G from u_k to $v_k = u_m$. This is a contradiction, since $u_k = v_l$ is greater than v_k in topological ordering. Therefore, any topological ordering coincides with P.

   Since G has a topological ordering, we conclude that this ordering is unique, because it coincides with P. By the same reason, G cannot have more than one Hamiltonian path. It follows that the set of Hamiltonian paths and the set of topological orderings both have exactly one element and these sets actually coincide.

2. Let's show that G has at least two different topological orderings. Consider some topological ordering $v_1, \dots, v_n$ of vertices of G. If $v_1, \dots, v_n$ is a path in G (i.e. all pairs of consecutive vertices $\{v_1, v_2\}, \dots, \{v_{n-1}, v_n\}$ are edges in G), then $v_1, \dots, v_n$ is a Hamiltonian path in G, which contradicts the fact that G has no Hamiltonian paths.

   So $v_1, \dots, v_n$ is not a path in G. It means that there exists a positive number i such that $\{v_i, v_{i+1}\}$ is not an edge in G. Then the sequence $v_1, \dots, v_{i-1}, v_{i+1}, v_i, v_{i+2}, \dots, v_n$ is a topological ordering which differs from $v_1, v_{i-1}, v_i, v_{i+1}, v_{i+2}, \dots, v_n$.

Exercise 2.5 *Eulerian tours.*

An *Eulerian tour* is a closed walk (Zyklus) that visits every edge exactly once.

In this exercise, we ask you to prove that a connected graph contains an Eulerian tour if and only if it does not contain a vertex of odd degree.

1. Prove that if a connected graph G contains an Eulerian tour, then G does not contain a vertex of odd degree.

   Solution: Consider an Eulerian tour $W = v_1, v_2, \dots, v_r, v_1$. Each entry of some vertex v_i to W corresponds to two edges adjacent to v_i (except the first or the last entry of v_1): There exist vertices v_a and v_b such that v_a, v_i, v_b is a segment in $v_1, v_2, \dots, v_r, v_1$, so the corresponding edges are $\{v_a, v_i\}$ and $\{v_i, v_b\}$. Notice that different entries of v_i correspond to disjoint pairs of edges. The first and the last entries of v_1 correspond to two edges $\{v_1, v_2\}$ and $\{v_r, v_1\}$.

   Since W is an Eulerian tour, for every vertex v each edge adjacent to v corresponds to some entry of v (or to the first and the last entries of v if $v = v_1$). It follows that the degree of any vertex v_i (except v_1) is $2t_i$, where t_i is a number of entries of v_i. The degree of v_1 is $2t_1 - 2$, where t_1 is a number of entries of v_1. Since G is connected, each vertex has an entry in W, so each vertex of G has even degree.

2. Prove that every connected graph without vertices of odd degree contains a Eulerian tour. Use mathematical induction on the number of edges.

   Hint: *Use the fact that every non-trivial connected graph without vertices of odd degree contains a cycle. Notice that this fact is a direct consequence of the fact that every non-trivial acyclic graph contains a leaf (which you proved in the previous exercise sheet).*

   - **Base Case.**
     Let $n = 0$. Then G has no edges. G does not contain odd degree vertices and has a trivial

Eulerian tour.

- **Induction Hypothesis.**
  Assume that the property holds for every positive integer $l \leq k$. That is, every connected graph G with $l \leq k$ edges such that each vertex in G has even degree has an Eulerian tour.

- **Inductive Step.**
  We must show that the property holds for $k+1$. Consider a connected graph $G = (V, E)$ with $k+1$ edges such that each vertex in G has even degree. Since G is not acyclic (otherwise it would have a leaf), G has a cycle $v_1, v_2 \ldots, v_r, v_1$.

  Let's remove all edges $E_0 = \{\{v_1, v_2\}, \ldots, \{v_{k-1}, v_r\}, \{v_r, v_1\}\}$ from G. This operation does not change parity of vertices: degrees of vertices $v_1, v_2 \ldots, v_r$ decrease by two and degrees of other vertices remain unchanged. Consider connected components $C_1 = (V_1, E_1), \ldots, C_m = (V_m, E_m)$ of the new graph. Each connected component C_i has less than $k+1$ edges and every vertex of C_i has even degree, so by induction hypothesis each C_i has an Eulerian tour W_i.

  Let's construct an Eulerian tour in G. We start with empty sequence. Then for all j from 1 to r we do the following: If $v_j \in C_t$ for some $1 \leq t \leq m$ which we did not meet before (i.e. there were no $v_i \in C_t$ for $i < j$), we add W_t to the sequence (we represent W_t starting with v_j; if $C_t = \{v_j\}$, we assume that $W_t = v_j$). Otherwise, if $v_j \in C_t$ for some $1 \leq t \leq m$ which we met before, we add v_j to the sequence. After this procedure we add v_1 to the sequence. Since all E_j for $0 \leq j \leq m$ are disjoint and $\cup_{j=0}^{m} E_j = E$, the constructed sequence is actually an Eulerian tour.

  By the principle of mathematical induction, the statement is true for connected graphs with any number of edges.

Submission: On Monday, 8.10.2018, hand in your solution to your TA *before* the exercise class starts.